Série 2a Solutions

Exercise 2a.1 – The strain gauge

A strain gauge is a resistive element (usually a piezoresistor) that is used to detect strain in a structure based on a change of its resistance as a function of strain. The gauge factor is defined as follows:

$$\text{Gauge factor: } GF = \frac{R_a - R_0}{R_0} \cdot \frac{1}{\varepsilon}$$

A force F is acting at the free end of a micro-bar. We measure the strain of the bar caused by this force using the resistance strain gauge sensor. We are applying a constant current (I) to the sensor and measuring the change in the voltage.

- The voltage measured on the strain sensor when $F = 0$ N is $V_0 = 2$ V
- Once $F \neq 0$ N, the voltage measured on the strain sensor is $V_a = 2.01$ V
- The micro-bar is made out of Silicon and has a Young's modulus of 150 GPa
- The strain gauge is made of doped Silicon and has a Gauge Factor of 30

Calculate the internal stress of the cantilever when the force is acting.

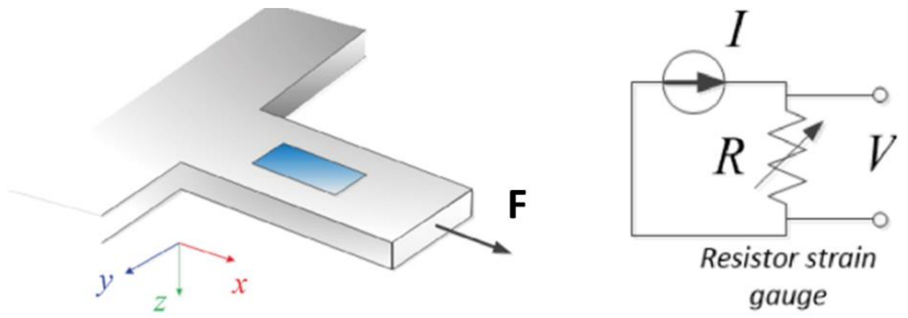


Figure 2a.1 | Strain-Gauge: (a) Device and (b) Equivalent circuit

Solution

Objective

We are looking for the internal stress in the micro-bar when the force is being applied

Given

The voltage measured when no force is applied: $V_0 = 2$ V

The voltage measured when there is an applied force: $V_F = 2.01$ V

The gauge factor

The current I is constant

The silicon Young's modulus

Formulas and principles

Hooke's law in 1D:

$$\sigma_x = E \cdot \varepsilon_x \tag{1}$$

where σ_x is the normal internal stress, ε_x is the axial strain and E is the Young's modulus of the material. We will combine this with the definition of the gauge factor

$$GF = \frac{R_F - R_0}{R_0} \cdot \frac{1}{\varepsilon_x} \tag{2}$$

where GF is the Gauge Factor, R_F is the gauge resistance when the force is applied and R_0 is the gauge resistance when no force is applied. To translate resistance to voltage we use Ohm's law

$$\frac{R_F - R_0}{R_0} = \frac{V_F - V_0}{V_0} \tag{3}$$

Calculations

Combining the equations, we have

$$\sigma_x = E \cdot \varepsilon_x = E \cdot \frac{R_F - R_0}{R_0} \cdot \frac{1}{GF} = E \cdot \frac{V_F - V_0}{V_0} \cdot \frac{1}{GF} \tag{4}$$

Numerical application

By replacing the parameters by their values, we obtain

$$\sigma_x = E \cdot \frac{V_F - V_0}{V_0} \cdot \frac{1}{GF} = 150 \text{ GPa} \cdot \frac{10^{-2} \text{ V}}{2 \text{ V}} \cdot \frac{1}{30} = 25 \text{ MPa} \tag{5}$$

Exercise 2a.2 - Shear modulus

Consider a 50 µm-thin film on a glass chip. The pattern is a rectangle of 0.5 mm width and 1 mm length. You submit the surface of the film to lateral shear stress (See Figure 2a.2). The shear force applied is 2.5 N. The shear strain measured is 0.01.

Calculate the shear modulus of the material

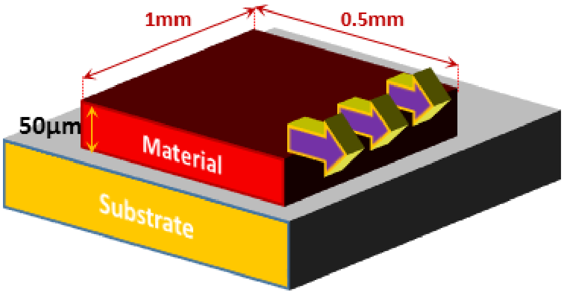
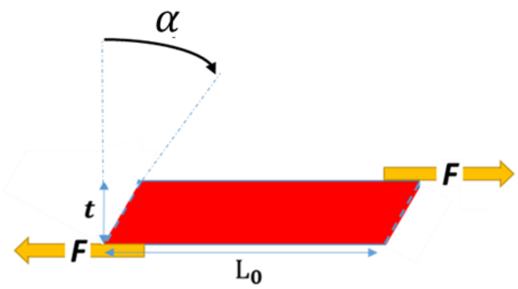


Figure 2a.2 | Thin film on a glass substrate.

Solution

Free Body Diagram for shear stress



Strain-Stress chart of the material

What is given?

Film length: $L_0 = 1$ mm
Film width: $W = 0.5$ mm
Film thickness: $t = 50$ µm
Shear force applied: $F = 2.5$ N
Shear strain: $\gamma = 0.01$

What is asked?

We are looking for the **shear modulus** of the material.

Principles and formulas

Shear strain (**no dimension**) under small angle approximation

$$\gamma = \tan(\alpha) \cong_{\alpha \to 0} \alpha \tag{1}$$

Hooke's law for shear stress

$$\tau = G \cdot \gamma \tag{2}$$

where G is the shear modulus of the material (Goal), γ is the angular deformation of the material in the shear direction and τ is the applied shear stress (unknown).

Definition for shear stress:

$$\tau = \frac{F}{A_s} \tag{3}$$

where F is the applied shear force *(known)*, and A_s is the area of the surface cross-section *(known)*.

Perform calculations in symbolic form

Using the equations, we obtain:

$$G = \frac{\tau}{\gamma} = \frac{F}{A_s} \cdot \frac{1}{\gamma} \tag{4}$$

State your answer

By replacing the parameters by their values, we obtain:

$$G = \frac{\tau}{\gamma} = \frac{F}{A_s} \cdot \frac{1}{\gamma} = \frac{2.5\ \text{N}}{10^{-3} \cdot 0.5 \cdot 10^{-3}\ \text{m}^2} \cdot \frac{1}{0.01} = 500\ \text{MPa} \tag{5}$$

Exercise 2a.3 – Stress and strain matrices

Consider a component of iron (E=20 GPa, v=0.3) loaded according to the following stress and strain matrices. The shear modulus (G) is equal to 7.7 GPa. One value in the stress matrix (σ_{xx}) and one value in the strain matrix (ε_{xy}) are not provided.

$$\hat{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_{xx} & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 0 \end{pmatrix} \mathrm{MPa} \qquad \hat{\boldsymbol{\varepsilon}} = \begin{pmatrix} 4 & \varepsilon_{xy} & 1.95 \\ \varepsilon_{xy} & -0.75 & 0 \\ 1.95 & 0 & -1.4 \end{pmatrix} * 10^{-4}$$

a) Solve by means of the compliance matrix the values for σ_{xx} and ε_{xy}

b) Verify your calculations by means of another method

Solution

a) Solving by means of the compliance matrix:

Normal stress:

$$\varepsilon_{xx} = \frac{1}{\mathrm{E}}\{\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\} \tag{1}$$

$$\mathrm{E} \cdot \varepsilon_{xx} + \nu\sigma_{yy} + \nu\sigma_{zz} = \sigma_{xx} \tag{2}$$

$$\sigma_{xx} = 20 * 10^{9} * 4 * 10^{-4} + 0.3 * (1 * 10^{6} + 0) = 8.3\ [\mathrm{MPa}] \tag{3}$$

Strain:

$$\gamma_{xy} = 2\varepsilon_{xy} = \frac{\tau_{xy}}{\mathrm{G}} \tag{4}$$

$$\varepsilon_{xy} = \frac{\tau_{xy}}{2\mathrm{G}} \tag{5}$$

$$\varepsilon_{xy} = \frac{2 * 10^{6}}{2 * 7.7 * 10^{9}} = 1.298 * 10^{-4} \approx 1.3 * 10^{-4} \tag{6}$$

b) Solving by means of the stiffness matrix:

Normal stress:

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\{(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}\} \tag{7}$$

$$\sigma_{xx} = \frac{20 * 10^{9}}{(1.3)(0.4)}\{0.7 * 4 * 10^{-4} - 0.3 * 0.75 * 10^{-4} - 0.3 * 1.4 * 10^{-4}\} \tag{8}$$

$$\sigma_{xx} = 8.288 * 10^{6} \approx 8.3\ [\mathrm{MPa}] \tag{9}$$

Strain:

$$\tau_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy} \tag{10}$$

$$\varepsilon_{xy} = \frac{\tau_{xy}}{2\mathrm{G}} \tag{11}$$

$$\varepsilon_{xy} = \frac{2 * 10^{6}}{2 * 7.7 * 10^{9}} = 1.298 * 10^{-4} \approx 1.3 * 10^{-4} \tag{12}$$

Exercise 2a.4 – Shear and normal strain in a shock mount

A shock mount, as shown in Figure 2a.4, is used to support a delicate instrument. The mount is composed out of a steel tube with inside diameter b, a central steel bar of diameter d that has an applied load F, and a hollow rubber cylinder (with thickness h) which is bonded to the tube and bar.

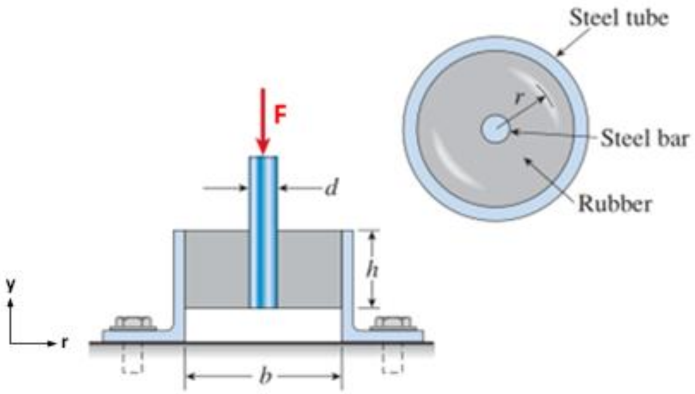


Figure 2a.4 | One dimensional sketch of the bar with a non-uniform cross-section

a) **Find an equation for the shear stress in terms of the radius (r)**
b) **Determine the shear strain in the rubber at the interface between the steel bar and the rubber**

Solution

a) Find an equation for the shear stress in terms of the radius (r)

We have to look at which plane the shear stress is applied. Since the force F is coming from the top the steel bar will push the rubber downwards. This means that the shear stress is applied at the interface between the steel rod and the rubber cylinder. The area of which is given as

$$A_{\parallel} = 2\pi r \cdot h \tag{1}$$

For the shear stress we know that the formula for applied shear stress is given as

$$\tau = \frac{N}{A_{\parallel}} \tag{2}$$

The equilibrium for this system, with respect to the internal force N, can be written as

$$-N - F = 0 \;\Rightarrow N = -F \tag{3}$$

Therefore, the applied shear stress, in terms of the radius is

$$\tau(r) = -\frac{F}{2\pi r h} \tag{4}$$

b) Determine the shear strain in the rubber at the interface between the steel bar and the rubber

The formula for the shear strain is given as

$$\gamma = \frac{\tau}{G} \tag{5}$$

Which gives the general shear strain equation to be

$$\gamma(r) = -\frac{F}{2\pi r h G} \tag{6}$$

The shear strain at the interface takes place at r = d/2 such that

$$\gamma\left(\frac{d}{2}\right) = -\frac{F}{\pi d h G} \tag{7}$$

Exercise 2a.5 - Material characterization

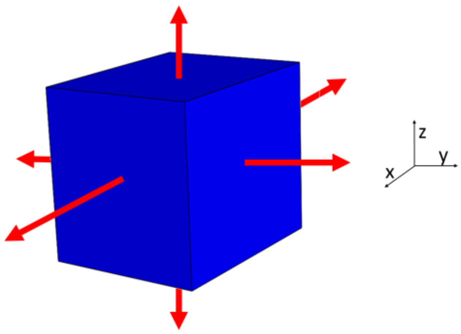


Figure 2a.5 | Cube under pull testing

We put a cube of 1 cm³ of this material (isotropic, homogeneous) under pull testing. Under the application of an external load, the measuring software provides us the following data for the material in its strain-stress elastic region:

$$\sigma_{xx} = 100 \text{ MPa};\ \sigma_{yy} = 100 \text{ MPa};\ \sigma_{zz} = 0 \text{ MPa}$$

$$\varepsilon_{xx} = 75 \cdot 10^{-5}; \varepsilon_{yy} = 75 \cdot 10^{-5}; \varepsilon_{zz} = -50 \cdot 10^{-5}$$

(a) Calculate the Young's modulus, the Poisson's ratio, and the Shear modulus of the material.

(b) Fill the compliance matrix for this material.

Solution

a) Calculate the Young's modulus, the Poisson's ratio, and the Shear modulus of the material.

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right) \\ \varepsilon_y &= \frac{1}{E}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right) \\ \varepsilon_z &= \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y)\right) \end{aligned} \tag{1}$$

Since we know that $\sigma_x = \sigma_y$, $\sigma_z = 0$, $\varepsilon_x = \varepsilon_y$ and $\varepsilon_z = -2\varepsilon_x/3$

$$\varepsilon_x = \frac{\sigma_x}{E}(1-\nu) \quad \text{and} \quad \varepsilon_z = -2\nu\frac{\sigma_x}{E} \Leftrightarrow \varepsilon_x = -\frac{3}{2}\varepsilon_z = 3\nu\frac{\sigma_x}{E} \tag{2}$$

Using this relation, we can equate both sides and solve

$$\begin{aligned} \varepsilon_x = \frac{\sigma_x}{E}(1-\nu) &= 3\nu\frac{\sigma_x}{E} \\ (1-\nu) = 3\nu &\Rightarrow \nu = \frac{1}{4} \end{aligned} \tag{3}$$

Then putting the value for Poisson's ratio back in one of the previous relations we can find E

$$E = -2\nu\frac{\sigma_x}{\varepsilon_z} \Rightarrow E = -2 * 0.25\frac{100 * 10^6}{-50 * 10^{-5}} = 100 * 10^9 = 100\ [\text{GPa}] \tag{4}$$

And finally we can calculate the shear modulus as

$$G = \frac{E}{2(1+\nu)} = \frac{100 * 10^9}{2(1+0.25)} = 40\ \text{GPa} \tag{5}$$

b) Fill the compliance matrix for this material.

$$\begin{pmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{pmatrix} \tag{6}$$

With the values we calculate for E, ν and G we find

$$\begin{aligned} \frac{1}{E} &= \frac{1}{100 * 10^9} = 10^{-11} \\ -\frac{\nu}{E} &= \frac{0.25}{100 * 10^9} = -0.25 * 10^{-11} \\ \frac{1}{G} &= \frac{1}{40 * 10^9} = 2.5 * 10^{-11} \end{aligned} \tag{7}$$

Filling in the matrix we find

$$\begin{pmatrix} 10^{-11} & -0.25 * 10^{-11} & -0.25 * 10^{-11} & 0 & 0 & 0 \\ -0.25 * 10^{-11} & 10^{-11} & -0.25 * 10^{-11} & 0 & 0 & 0 \\ -0.25 * 10^{-11} & -0.25 * 10^{-11} & 10^{-11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 * 10^{-11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.5 * 10^{-11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.5 * 10^{-11} \end{pmatrix} \tag{8}$$

OPTIONAL - Exercise 2a.6 - Strain matrix and vector

For a certain body, a displacement field is given as

$$u(x, y, z) = -\alpha yz, \qquad v(x, y, z) = \alpha yz, \qquad w(x, y, z) = 0$$

a) **Determine the strain tensor which belongs to this displacement field**
b) **Show the components that relate to the normal and shear strains**

Solution

Given:

3D displacement field

Solution:

a) Determine the strain tensor which belongs to this displacement field

First, we write down the definition of a strain tensor

$$\varepsilon = \begin{bmatrix} \varepsilon_x & 0.5 \cdot \gamma_{xy} & 0.5 \cdot \gamma_{xz} \\ 0.5 \cdot \gamma_{xy} & \varepsilon_y & 0.5 \cdot \gamma_{yz} \\ 0.5 \cdot \gamma_{xz} & 0.5 \cdot \gamma_{yz} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & 0.5\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & 0.5\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ 0.5\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & 0.5\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ 0.5\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & 0.5\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix} \quad (1)$$

Now solve for the partial derivatives for each component of the strain tensor

$$\begin{gathered} \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = -\alpha z, \quad \frac{\partial u}{\partial z} = -\alpha y \\ \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = \alpha z, \quad \frac{\partial v}{\partial z} = \alpha y \\ \frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = 0 \end{gathered} \quad (2)$$

Inserting those partial derivatives in the strain tensor leads to:

$$\varepsilon = \begin{bmatrix} 0 & -\frac{\alpha z}{2} & -\frac{\alpha y}{2} \\ -\frac{\alpha z}{2} & \alpha z & \frac{\alpha y}{2} \\ -\frac{\alpha y}{2} & \frac{\alpha y}{2} & 0 \end{bmatrix} = \frac{\alpha}{2}\begin{bmatrix} 0 & -z & -y \\ -z & 2z & y \\ -y & y & 0 \end{bmatrix} \quad (3)$$

b) Show the components that relate to the normal and shear strains

Taking the diagonal of strain matrix, we find the normal strains as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha z \\ 0 \end{bmatrix} \quad (4)$$

And from the offset indices we find the shear strains as

$$\begin{bmatrix} \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} -\alpha z \\ -\alpha y \\ \alpha y \end{bmatrix} \quad (5)$$